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Junction Conditions and Static Fluid Cylinders in Lyra's Geometry

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Junction conditions for Sen's theory in Lyra's geometry are considered. It is proposed that for any gauge function the standard O'Brien–Synge and Lichnerowicz junction conditions should be supplemented by demanding continuity of the displacement vector across the interface. A class of internal solutions of the Sen equations with a source term given by the energy-momentum tensor of a one-component perfect fluid with the ultrarelativistic equation of state that is expressible in terms of Bessel functions is proposed. The internal solution is regularly matched by means of the junction conditions to the exterior solution. The resulting two-parameter solution is globally non-Euclidean.

1. INTRODUCTION

Modifications and extensions of general relativity have intrigued theorists for a long time. One of the most important motivations for studying this group of problems is the attempt to construct reasonable cosmological models which, while possessing desired features such as symmetry and physical characteristics of the fields and matter, are free from the puzzles and singularities of standard cosmology.

Einstein himself very early considerably changed the original 1915 gravitational field equations by adding a so-called "cosmological term," which since that time has periodically appeared and disappeared in physics, having its most spectacular comeback in the inflationary model.

Over the past 70 years many studies have been carried out to fulfill the great vision which has its roots in the works of Riemann and Clifford—to unify all interactions in a purely geometrical way. Among them, one that

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appeared almost simultaneously with the Einstein theory (Weyl, 1918) received considerable attention. Although Weyl's efforts to unify gravitational and electromagnetic phenomena by means of the fundamental changes of the Riemannian geometry missed the target and were early recognized as unsatisfactory, his approach contains deep and fruitful ideas: the beginnings of the general theory of connections and the conception of a gauge invariance.

On the basis of an interesting generalization of the Riemannian geometry, which may be also considered as the modification of the Weyl geometry, invented by Lyra (1951) to overcome the nonintegrability of length transfers, a generic and principal difficulty of the Weyl theory, Sen (1957) constructed an analog of the Einstein gravitational field equations. In the Lyra geometry Weyl's concept of "*Eichinvarianz*," which is a purely metrical concept, is modified by introducing a gauge function into the structureless manifold. Roughly speaking, in a Lyra geometry one introduces another type of covariance; the gauge covariance and the choice of a gauge function as well as the choice of coordinate system are arbitrary. It should be noted, however, that unlike the Weyl geometry, in the Lyra geometry tensors (treated as multilinear mappings) transform under the action of gauge transformations with zero weight, while the base vectors and one-forms transform with the weight 1 and -1, respectively.

One method of constructing global solutions in general relativity is to match two solutions that correspond to two different physical situations. The junction conditions in general relativity have been subjected to exhaustive analysis by a number of workers (O'Brien and Synge, 1952; Lichnerowicz, 1955; Israel, 1958, 1966; Synge, 1960; Nariai, 1965). Unfortunately, although the Sen theory and its more recent generalizations (Sen, 1968; Sen and Vanstone, 1972) have received considerable attention in the cosmological context and although many interesting solutions both with and without a source term have been presented and studied (Sen, 1957; Halford, 1970; Bhamra, 1974; Beesham, 1986*a*,*b*, 1988; Ram and Singh, 1992; Singh and Agrawal, 1992; Singh and Singh, 1991, 1992), little is known about the global solutions obtained by means of matching the internal solution to the external one [see, however, Matyjasek and Rogatko (1992)]. It would be fair to say that this is a common situation in alternative theories. It is interesting therefore to examine more closely this group of problems.

In this paper we shall consider boundary conditions at a 3-space of discontinuity (interface) in Lyra geometry in the spirit of O'Brien and Synge and of Lichnerowicz and subsequently illustrate the procedure in the particular problem of static fluid distributions that have reasonable physical characteristics in a space possessing cylindrical symmetry.

The paper is organized as follows: in Section 2 we present the necessary mathematical background; the more complicated formulas are relegated to Appendix A. Since here we are addressing a specific group of problems, it should be noted that our definitions of the Lyra differentiable manifold and structures constructed on it are weaker than those usually encountered in the literature. Section 3 deals with the Sen equations in the normal gauge. The projective structure is investigated in Section 4. Boundary conditions and the proof of admissibility of an analog of the Gauss coordinates are presented in Section 5. The application of the O'Brien–Synge and Lichnerowicz junction conditions to the problem of static fluid cylinders is analyzed in Section 6. In Appendix B the Sen equations that are valid for any gauge function are derived from the variational principle. The same equations are obtained by "transforming back" the normal gauge Sen equations, which means that they are indeed doubly covariant. The relation to the conformally transformed Einstein field equations is also considered.

Finally, this paper corrects some earlier errors that one may encounter in the literature on the subject.

2. LYRA GEOMETRY

To begin we collect some basic information concerning the Lyra geometry that will be useful later. Lyra C^∞ manifolds are treated in Sen and Dunn (1971) and Sen and Vanstone (1972).

We assume the space M to be a connected second countable Hausdorff space. An essential notion in the Lyra formulation is a *reference system*. The reference system on M is a triple (U_i, ψ_i, χ_i), where:

1. U_i is an open subset of M.
2. ψ_i is a homeomorphism of U_i onto an open subset of R^n.
3. $\chi_i\colon U_i \to R\backslash\{0\}$ is a gauge function.

A C^k Lyra manifold is M with a collection of reference systems that cover M and such that the maps $\psi_i \circ \psi_j^{-1}$ and gauge functions $\chi_i \circ \psi_i^{-1}$ are C^k.

A C^k differentiable curve on a Lyra manifold is a mapping $\gamma\colon I \subset R \to M$ such that $\psi \circ \gamma\colon R \to R^n$ is C^k. One can introduce a tangent vector in the usual manner. If we assume that the homeomorphism ψ is given by functions x^μ, the natural basis for a tangent space is given by $e_\mu = \chi^{-1}\, \partial/\partial x^\mu$ rather than $\partial/\partial x^\mu$, so that the tangent vector $X \in T_p$ can be written as $X = X^\mu e_\mu$.

Elements of the cotangent space T_p^* are linear functions from $T_p \to R$. Basis 1-forms are given by the condition $e^\mu(e_\nu) = \delta^\mu_\nu$ and hence $e^\mu = \chi\, dx^\mu$.

The inner product

$$\langle \cdot , \cdot \rangle : T_p^* \times T_p \to R \tag{1}$$

is defined by

$$\langle \omega | v \rangle = \omega(v) = \omega_\mu v^\mu \tag{2}$$

A tensor of type (r, s) is a multilinear mapping of r copies of T_p^* and s copies of T_p into R:

$$T: \quad T_p^* \times T_p^* \times \cdots \times T_p^* \times T_p \times T_p \times \cdots \times T_p \to R \tag{3}$$

Let $T_{s,p}^r$ denotes the set of type (r, s) tensors. An element t of $T_{s,p}^r$ may be exhibited as

$$t = t_{\mu_1 \cdots \mu_r}^{\nu_1 \cdots \nu_s} e^{\mu_1} \otimes \cdots \otimes e^{\mu_l} \otimes e_{\nu_1} \otimes \cdots \otimes e_{\nu_k} \tag{4}$$

Under the transformation to a new reference system $(\tilde{U}, \tilde{\psi}, \tilde{\chi})$ the components of a tensor of type (r, s) transform according to

$$\tilde{t}_{\nu_1 \cdots \nu_s}^{\mu_1 \cdots \mu_r} = \lambda^{r-s} \frac{\partial \tilde{x}^{\mu_1}}{\partial x^{\gamma_1}} \cdots \frac{\partial \tilde{x}^{\mu_r}}{\partial x^{\gamma_r}} \frac{\partial x^{\delta_1}}{\partial \tilde{x}^{\nu_1}} \cdots \frac{\partial x^{\delta_s}}{\partial \tilde{x}^{\nu_s}} t_{\delta_1 \cdots \delta_s}^{\gamma_1 \cdots \gamma_r} \tag{5}$$

where

$$\lambda = \frac{\tilde{\chi}}{\chi} \tag{6}$$

A C^r linear connection ∇ on the C^k Lyra manifold $(k \geq r + 2)$ is a mapping $X \to \nabla X$ from the C^{k-1} vector field X to a C^{k-2} tensor field of type $(1, 1)$ such that

$$\nabla(X + Y) = \nabla X + \nabla Y \tag{7}$$

$$\nabla f X = df \otimes X + f \nabla X \tag{8}$$

The C^r $(r \leq k - 1)$ metric tensor g is a symmetric tensor field of type $(0, 2)$ on M, and in the natural basis it has the following form:

$$g = g_{\mu\nu} e^\mu \otimes e^\nu \tag{9}$$

In the further applications we shall require the Lyra manifold to be C^4 except at interfaces, where class C^2 is assumed. Therefore the metric tensor is taken to be C^3 and only C^1 at the interface.

There is a unique Lyra connection ${}^{\mathrm{L}}\nabla_X$ defined as

$${}^{\mathrm{L}}\nabla_X Y = \langle \nabla Y, X \rangle \tag{10}$$

satisfying condition of metricity

$${}^{\mathrm{L}}\nabla_X g = 0 \tag{11}$$

and with a torsion $T(X, Y)$ given by the following expression:

$$T(X, Y) = {}^{L}\nabla_Y X - {}^{L}\nabla_X Y - [X, Y] = \frac{1}{2}\{\langle\omega, X\rangle Y - \langle\omega, Y\rangle X\} \tag{12}$$

where ω is a given Lyra 1-form and $[X, Y]$ is a Lie bracket of X and Y (see Appendix A). In a local reference system the Lyra connection is specified by functions ${}^{L}\Gamma^{\alpha}_{\beta\gamma}$ defined as

$${}^{L}\nabla_{e_\beta} e_\gamma = {}^{L}\Gamma^{\alpha}_{\gamma\beta} e_\alpha \tag{13}$$

It can be easily shown that the Lyra connection is

$${}^{L}\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{\chi}\, {}^{R}\Gamma^{\alpha}{}_{\beta\gamma} + \frac{1}{2}(\delta^{\alpha}_{\gamma}\phi_\beta - g_{\beta\gamma}\phi^{\alpha}) \tag{14}$$

where ${}^{R}\Gamma^{\alpha}_{\beta\gamma}$ is a Riemann connection and

$$\phi = \omega + d \ln \chi^2. \tag{15}$$

The quantity ϕ is the *displacement vector*. The term $d \ln \chi^2$ under the change of the gauge function transforms according to the rule

$$d \ln \tilde{\chi}^2 = d \ln \lambda^2 + d \ln \chi^2 \tag{16}$$

and consequently the displacement vector ϕ is not a Lyra 1-form. From (15) and (16) one readily has

$$\tilde{\phi}_\mu = \frac{1}{\lambda}\left(\phi_\mu + \frac{1}{\chi}\frac{\partial \ln \lambda^2}{\partial x^\mu}\right) \tag{17}$$

From the law of transplantation of base vectors and (13) one concludes that under change of reference system from (U, ψ, χ) to $(\tilde{U}, \tilde{\psi}, \tilde{\chi})$ the connection coefficients transform as follows:

$${}^{L}\Gamma^{\alpha}_{\beta\gamma} = \lambda^{-1} A^{\alpha}_{\mu} A^{\nu}_{\beta} A^{\tau}_{\gamma}\, {}^{L}\Gamma^{\mu}_{\nu\tau} + \chi^{-1} A^{\mu}_{\rho}(A^{\rho}_{\beta})_{,\gamma} - \frac{1}{2}\chi^{-1}\delta^{\alpha}_{\beta} A^{\rho}_{\gamma}(\ln \lambda^2)_{,\rho} \tag{18}$$

where a comma denotes partial differentiation

$$A^{\mu}{}_{\alpha} = \frac{\partial x^\mu}{\partial x^\alpha} \tag{19}$$

It is said that a vector Y is Lyra transported along a curve γ with a tangent vector X if

$${}^{L}\nabla_X Y = 0 \tag{20}$$

By (11) it follows then that the length transfers are integrable and consequently the most serious objection to Weyl geometry is removed. Since the Lyra manifold is endowed with connection ${}^{L}\nabla$, it is possible to introduce a curvature operation ρ defined as

$$\rho(X, Y) = {}^{\mathrm{L}}\nabla_X {}^{\mathrm{L}}\nabla_Y - {}^{\mathrm{L}}\nabla_Y {}^{\mathrm{L}}\nabla_X - {}^{\mathrm{L}}\nabla_{[X,Y]} \tag{21}$$

The curvature tensor is 4-linear mapping attaching to every 1-form ω and vectors X, Y, Z a number according to the rule

$${}^{\mathrm{L}}R(\omega, Z, X, Y) = \langle \omega, \rho(X, Y)Z \rangle \tag{22}$$

In a local reference system with a natural base vectors and cobase forms the mapping R has the following form:

$${}^{\mathrm{L}}R(\omega, Z, X, Y) = {}^{\mathrm{L}}R^{\alpha}{}_{\beta\gamma\delta} e_\alpha \otimes e^\beta \otimes e^\gamma \otimes e^\delta(\omega, Z, X, Y) \tag{23}$$

and

$${}^{\mathrm{L}}R^{\alpha}{}_{\beta\gamma\delta} = \chi^{-2}\left(\frac{\partial\chi\, {}^{\mathrm{L}}\Gamma^{\alpha}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\chi\, {}^{\mathrm{L}}\Gamma^{\alpha}_{\beta\gamma}}{\partial x^\delta}\right) + {}^{\mathrm{L}}\Gamma^{\alpha}_{\eta\gamma}\, {}^{\mathrm{L}}\Gamma^{\eta}_{\beta\delta} - {}^{\mathrm{L}}\Gamma^{\alpha}_{\eta\delta}\, {}^{\mathrm{L}}\Gamma^{\eta}_{\beta\gamma} \tag{24}$$

The contracted curvature tensor (an analog of a Ricci tensor) is obtained by setting $\alpha = \delta$ in (24),

$${}^{\mathrm{L}}R^{\alpha}{}_{\beta\gamma\alpha} = {}^{\mathrm{L}}R_{\beta\gamma} \tag{25}$$

The curvature scalar is

$${}^{\mathrm{L}}R = g^{\alpha\beta}\, {}^{\mathrm{L}}R_{\alpha\beta} = \frac{{}^{\mathrm{R}}R}{\chi^2} + \frac{3}{\chi}\, {}^{\mathrm{R}}\nabla_\alpha \phi^\alpha + \frac{3}{2}\phi_\alpha\phi^\alpha + \frac{3}{2}\Phi_\alpha\phi^\alpha \tag{26}$$

where

$$\langle d \ln \chi^2 | e_\alpha \rangle = \Phi_\alpha = \chi^{-1}\frac{\partial \ln \chi^2}{\partial x^\alpha} \tag{27}$$

and the superscript $^{\mathrm{R}}$ refers to the Riemannian quantities. Φ_μ transforms exactly as ϕ_μ and their difference is a Lyra tensor.

3. THE SEN EQUATIONS

In the Lyra geometry one introduces another type of covariance, namely the gauge covariance. Since every equality of Lyra tensors may be easily converted into equality of the gauge-invariant quantities, one can speak equally well of the gauge invariance. Indeed, it suffices to multiply a Lyra (p, q) tensor expressed in the gauge χ by a factor x^{q-p}. In what follows we shall refer to the gauge covariance rather than invariance. Since the choice of the gauge function is entirely optional, we shall work with the so-called normal gauge, i.e., $\chi = 1$.

On the basis of the Lyra geometry Sen (1957) constructed a theory of gravitation whose equations are the consequence of the variational principle

$$\delta(I + J) = 0 \tag{28}$$

where

$$I = \int {}^{\mathrm{L}}R\sqrt{-g}\, d^4x \tag{29}$$

$$J = \int \Lambda\sqrt{-g}\, d^4x \tag{30}$$

Λ is the Lagrangian density of matter and the variation is taken with respect to the metric tensor only. The variational principle yields

$$G_{\mu\nu} + \frac{3}{2}\phi_\mu\phi_\mu - \frac{3}{4}g_{\mu\nu}\phi_\sigma\phi^\sigma = -8\pi T_{\mu\nu} \tag{31}$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein and the energy-momentum tensor, respectively. The displacement vector has no clear and unambiguous physical interpretation; however, the normal gauge Sen equations closely resemble the equations of Hoyle and Narlikar (1948).

Soleng (1987) has pointed out that cosmological models constructed within the framework of the Lyra geometry with constant displacement vector are either analogous to the theory with creation field or are equivalent to the standard cosmologies with nonzero cosmological constant and with a special vacuum field.

The vacuum Sen equations may be rewritten in the following form:

$$R_{\mu\nu} = -\frac{3}{2}\phi_\mu\phi_\nu \tag{32}$$

If we allow the displacement vector to have a form $(0, \beta(r), 0, 0)$ as we shall do in what follows because of cylindrical symmetry, equation (32) may be rewritten as

$$R_{\mu\nu} = \frac{3}{2}\beta^2\delta^r_\mu\delta^r_\nu g_{rr} \tag{33}$$

One may therefore formally regard the displacement vector as a quantity that generates an analog of the cosmological term that enters the equations in an asymmetric way. When the displacement vector is taken to be constant the analogy is even more transparent.

Varying the action integral (28) with respect to the displacement vector yields $\phi_\mu = 0$, and therefore one has (in a normal gauge) just the

Einstein field equations. This and related problems are discussed in Appendix B.

4. THE PROJECTIVE STRUCTURE

A curve γ on a Lyra manifold is said to be a geodesic if the tangent vector to γ transported parallely in the Lyra connection remains a multiple to itself. From this definition one has

$$\chi \frac{d^2x^\mu}{ds^2} + \chi^2\, {}^{\mathrm{L}}\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} + \frac{1}{2}\chi^2\Phi_\alpha \frac{dx^\alpha}{ds}\frac{dx^\mu}{ds} = \chi\psi\frac{dx^\alpha}{ds} \tag{34}$$

where ψ is a proportionality factor. By suitable reparametrization the equation for a geodesic may be reduced to the affinely parametrized form:

$$\chi \frac{d^2x^\mu}{dt^2} + \chi^2\, \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \frac{1}{2}\chi^2\Phi_\alpha \frac{dx^\alpha}{dt}\frac{dx^\mu}{dt} = 0 \tag{35}$$

It should be noted that extremal curves obtained from the Fermat principle

$$\frac{d^2x^\mu}{ds^2} + \left[{}^{\mathrm{R}}\Gamma^\mu_{\alpha\beta} + \frac{\chi}{2}(\delta^\mu_\alpha\Phi_\beta + \delta^\mu_\beta\Phi_\alpha - g_{\alpha\beta}\Phi^\mu)\right]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{36}$$

do not coincide with the autoparallels, though they may be regarded, as shown by an inspection of equation (14), as autoparallels of the Lyra connection with vanishing 1-form ω.

Now let us consider when two Lyra connections ${}^{\mathrm{L}}\nabla$ and ${}^{\mathrm{L}}\hat{\nabla}$ have the same geodesics with possible different parametrizations. Such connections are said to be projectively equivalent.

We have the analog of the following theorem originally due to Weyl (1921): two connections ${}^{\mathrm{L}}\nabla$ and ${}^{\mathrm{L}}\hat{\nabla}$ are projectively equivalent if and only if there is unique Lyra 1-form Ω such that

$$S(X, Y) = \langle\Omega|X\rangle Y + \langle\Omega|Y\rangle X \tag{37}$$

where $S(X, Y) = \frac{1}{2}({}^{\mathrm{L}}\hat{\nabla}_X Y + {}^{\mathrm{L}}\hat{\nabla}_Y X - {}^{\mathrm{L}}\nabla_X Y - {}^{\mathrm{L}}\nabla_Y X)$.

This theorem may be easily proved by a slight modification of arguments of the Riemannian case (Spivak, 1979).

In a local reference system two symmetric Lyra connections[2] are projectively equivalent iff

$${}^{\mathrm{L}}\hat{\Gamma}^\alpha_{\beta\gamma} = {}^{\mathrm{L}}\Gamma^\alpha_{\beta\gamma} + \delta^\alpha_\beta\Omega_\gamma + \delta^\alpha_\gamma\Omega_\beta \tag{38}$$

[2]Studying geodesics, it is sufficient to consider the symmetric connections only.

It should be emphasized that the difference of the connections is a Lyra tensor. Inspection of equations (35) and (36) shows that connections ${}^{L}\Gamma^{\mu}_{\alpha\beta}$ and $\chi^{-1}\,{}^{R}\Gamma^{\mu}_{\alpha\beta}+\frac{1}{2}(\delta^{\mu}_{\beta}\Phi_{\alpha}-g_{\alpha\beta}\Phi^{\mu})$ are not projectively equivalent unless an additional condition is fulfilled:

$$g(X, X)=0 \tag{39}$$

where X is the tangent vector to geodesics. Since the Sen theory does not specify which curve follows a massive body, this theory should be in that sense considered as incomplete. From (35) and (36) one readily concludes that when $\omega=0$ Fermat geodesics coincide with autoparallels.

5. BOUNDARY CONDITIONS

As we have said, we assume the Lyra manifold to be piecewise of class C^4 and of class C^2 at the interface Σ. We call admissible any reference system belonging to this very class. Let Σ be a smooth 3-space defined by equations

$$x^{\mu}=f^{\mu}(\xi^1, \xi^2, \xi^3) \tag{40}$$

where ξ are the coordinates on the interface. Now, for any chosen gauge function we can construct a reference system, being in fact an analog of the Gauss geodesic coordinates, taking the $\tilde{x}^0$ coordinate to be the unique parameter τ on each of the Fermat geodesics such that $\tau=0$ on Σ and putting $\tilde{x}^i=\xi^i$, where $i=1, 2, 3$. Let y^{μ} compose admissible coordinates. From equation (36) one has the expansion

$$y^{\mu}=f^{\mu}_{|p}+\tau T^{\mu}_{|p}-\frac{1}{2}\tau^2(G^{\mu}_{\alpha\beta}T^{\alpha}T^{\beta})_{|p}+\cdots \tag{41}$$

where

$$T^{\mu}=\frac{dy^{\mu}}{d\tau} \tag{42}$$

$$G^{\mu}_{\alpha\beta}={}^{R}\Gamma^{\mu}_{\alpha\beta}+\frac{\chi}{2}(\delta^{\mu}_{\alpha}\Phi_{\beta}+\delta^{\mu}_{\beta}\Phi_{\alpha}-g_{\alpha\beta}\Phi^{\mu}) \tag{43}$$

and the subscript p indicates the point of evaluation. Since the metric tensor and the gauge function are taken to be C^1, the above expansion is valid up to a second order, i.e., the terms written explicitly are independent of the side of 3-space Σ that we approach as $\tau\to 0$. It is evident therefore that the transformation between y^{μ} and $\tilde{x}^{\mu}$ is C^2 and the components of the metric tensor expressed in Gauss coordinates as well as coefficients of the Lyra connection are continuous across Σ. Therefore we have proved that for any gauge function the Gauss coordinates (in the above sense) are admissible.

In the Lichnerowicz approach we start by introducing admissible coordinates such that the interface is specified by $x^0 = 0$, where x^0 may be any of x^μ. Then the O'Brien–Synge junction conditions may be written as

$$[g_{\mu\nu}]_1 = [g_{\mu\nu}]_2 \tag{44a}$$

$$\left[\frac{\partial g_{ij}}{\partial x^0}\right]_1 = \left[\frac{\partial g_{ij}}{\partial x^0}\right]_2 \tag{44b}$$

$$[T_\mu{}^0]_1 = [T_\mu{}^0]_2 \tag{44c}$$

Here the symbol $[\cdot]$ denotes the boundary value of any quantity at the interface between two adjacent regions, i.e., $[B]_{1(2)} = \lim_{x^0 \to 0-(+)} B$. In the general theory of relativity the O'Brien–Synge conditions are not independent. Let us therefore examine the junction conditions in the Lyra geometry more closely. Let us supplement the O'Brien–Synge boundary conditions expressed in the admissible coordinates; we shall call them O'Brien–Synge–Lichnerowicz boundary conditions in the Lyra geometry by demanding continuity of the displacement vector, i.e.,

$$[\phi_\beta]_1 = [\phi_\beta]_2 \tag{44d}$$

Since the gauge functions are supposed to be C^2 at the interface, whereas the Lyra 1-forms ω are only C^0, the conditions (44a)–(44d) are not independent. Indeed, by equation (B.3) it can be easily verified that if equations (44a), (44b), and (44d) are satisfied, so is equation (44c). In the normal gauge condition, (44d) leads to the equality of the Lyra forms ω. It should be noted that though the same line element is associated with the displacement vectors ϕ and $-\phi$, the junction conditions remain unchanged.

6. STATIC FLUID CYLINDERS

Now we apply the foregoing considerations to the particular problem of the cylindrically symmetric solution to the Sen equations. The metric tensor that possesses the desired symmetry is

$$g = e^{2\nu}\, dt \otimes dt - dr \otimes dr - e^{2\lambda}\, d\phi \otimes d\phi - e^{2\mu}\, dz \otimes dz \tag{45}$$

where μ, λ, and μ are functions of r. The Sen equations with a source term given by the hydrodynamic energy-momentum tensor

$$T = T_{\mu\nu} e^\mu \otimes e^\nu \tag{46}$$

where

$$T_{\nu}^{\mu} = \text{diag}(-p, -p, -p, \rho) \tag{47}$$

and p and ρ are pressure and energy density, respectively, may be written as follows:

$$\lambda'\mu' + \mu'\nu' + \nu'\lambda' - \frac{3}{4}\beta^2 = 8\pi p \tag{48a}$$

$$\mu'' + \nu'' + \mu'^2 + \mu'\nu' + \nu'^2 + \frac{3}{4}\beta^2 = 8\pi\rho \tag{48b}$$

$$\lambda'' + \nu'' + \lambda'^2 + \lambda'\nu' + \nu'^2 + \frac{3}{4}\beta^2 = 8\pi\rho \tag{48c}$$

$$\lambda'' + \mu'' + \lambda'^2 + \lambda'\mu' + \nu'^2 + \frac{3}{4}\beta^2 = -8\pi\rho \tag{48d}$$

where prime denotes differentiation with respect to coordinate r. Following the method propounded by Evans (1977), the Sen equations (48a)–(48d) may be rewritten in the following simpler form in the present context:

$$\mu'' - \xi'\mu' + 3(\xi'' + e^{-2\xi} + 3\beta^2)u = 0 \tag{49}$$

$$4\pi(5p - \rho) = (e^{\xi})''e^{-\xi} \tag{50}$$

$$32\pi p = (\xi' + 3\mu')(\xi' - \nu') - e^{-2\xi} - 3\beta^2 \tag{51}$$

with the functions u and ξ connected with the metric potentials by

$$\xi = \lambda + \mu + \nu \tag{52a}$$

$$\eta = \lambda - \mu \tag{52b}$$

$$\eta' = e^{-\xi} \tag{52c}$$

$$u = e^{3\nu} \tag{52d}$$

Since we have freedom in the choice of any two of the relevant functions of the problem, in the latter we shall put $\xi = \ln r$, which considerably simplifies equation (49). Therefore we are looking for a regular solution expressible in terms of known transcendental functions that lead to a physically reasonable energy-momentum tensor and, in view of further applications, we demand the radial pressure to vanish for a definite value of r.

It can be easily shown that for $\beta^2 = k^2q^2r^{2q-2}$, the equation admits simple solutions in terms of Bessel functions. As an example, we consider more closely the case $q = 2$ with the displacement vector given by $\beta^2 = (4/9)\lambda^2r^2$, where the numerical factor 4/9 is introduced for convenience.

One therefore has

$$u = rC\left(\frac{2}{3}\lambda r^2\right) \tag{53}$$

where C stands for Bessel functions of a half order. Hence the line element that fulfills all the mentioned requirements has the following simple form:

$$ds^2 = f^{2/3}\, dt^2 - dr^2 - f^{-1/3}(r^2\, d\phi^2 + dz^2) \tag{54}$$

where $f = \cos \lambda r^2 + A \sin \lambda r^2$. The pressure therefore is

$$32\pi p = \frac{4}{3}[\lambda\psi - \lambda^2 r^2(1 + \psi^2)] \tag{55}$$

with

$$\psi = \frac{a \cos \lambda r^2 - \sin \lambda r^2}{\cos \lambda r^2 + A \sin \lambda r^2} \tag{56}$$

It should be noted that $p(0) = A\lambda/24\pi$, and therefore $A\lambda > 0$.

Putting in the r.h.s. of (48a)–(48d) $p = \rho = 0$ and solving the system of resulting equations, one obtains the external vacuum solution, which is found to be (Matyjasek and Rogatko, 1992)

$$\begin{aligned} g = {} & P^2(r + r_0)^{a_1}\, dt \otimes dt - dr \otimes dr - Q^2(r + r_0)^{a_2}\, d\phi \otimes d\phi \\ & - R^2(r + r_0)^{a_3}\, dz \otimes dz \end{aligned} \tag{57}$$

with the following constraints on the constants:

$$a_1 + a_2 + a_3 = 2 \tag{58a}$$

and

$$a_1 a_2 + a_2 a_3 + a_1 a_3 = 3\eta^2 \tag{58b}$$

The displacement vector in this case has the form

$$\beta = \frac{\eta}{r + r_0} \tag{59}$$

It can be easily shown from the definition of the analog of the Gauss coordinates in Lyra geometry that the cylindrical coordinates are admissible and therefore the O'Brien–Synge–Lichnerowicz junction conditions comprise the continuity of the metric tensor and its first derivatives with respect to the r coordinate and continuity of the radial pressure. These conditions should be supplemented by requirement of continuity of the displacement vector across the interface. The relevant equations therefore have the form

$$P^2(b + r_0)^{a_1} = f^{2/3}(b) \tag{60a}$$

$$Q^2(b + r_0)^{a_2} = f^{-1/3}(b) \tag{60b}$$

$$R^2(b + r_0)^{a_3} = f^{-1/3}(b) \tag{60c}$$

$$a_1(b + r_0)^{-1} = \frac{4}{3} b\lambda\psi(b) \tag{61a}$$

$$a_2(b + r_0)^{-1} = \frac{2}{b} - \frac{2}{3} b\lambda\psi(b) \tag{61b}$$

$$a_3(b + r_0)^{-1} = -\frac{2}{3} b\lambda\psi(b) \tag{61c}$$

$$\frac{\eta}{b + r_0} = \frac{2}{3} \lambda b \tag{62}$$

The complete set of equations required for matching consists of (58a), (58b), (60a)–(62), and

$$p(b) = 0 \tag{63}$$

In order for equations (61a)–(61c) to be compatible with equation (58a) one must set $r_0 = 0$. Substituting (61a)–(61c) into (58b) and making use of (63), one has

$$\frac{4}{9} \lambda^2 b^2 = \frac{\eta^2}{b^2} \tag{64}$$

and therefore this equation yields no more informations than equation (62). Consequently we have eight equations for ten constants. We may choose two of them, say λ and $p(0)$, as parameters. This situation resembles the solutions studied by Marder (1958) and Bonnor (1979, 1982) in the context of general relativity and shows another important feature of globally non-Euclidean topology. The ratio of the proper circumference to the proper radius of the circle in the $t = \text{const}$, $z = \text{const}$ slice, $\epsilon(r)$, in the exterior metric is given by

$$2\pi Q r^{a_2/2 - 1} \tag{65}$$

It should be stressed that though the matching conditions are not independent, we kept the full set of them for convenience. This dependence manifests itself explicitly in equations (62) and (64).

APPENDIX A

A differential k-form on the Lyra manifold in the natural basis has the form

$$\Omega = \omega_{\mu_1 \cdots \mu_k} e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \tag{A1}$$

The exterior derivative of the k-form defined in such a way as to be doubly covariant may be written in the following form:

$$d\Omega = \left(D_\nu \omega_{\mu_1 \cdots \mu_k} + \frac{k}{2} \Phi \omega_{\mu_1 \cdots \mu_k} \right) e^\nu \wedge e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \tag{A2}$$

where $D_\nu = (1/\chi)\, \partial/\partial x^\nu$. It could be easily shown that $dd\Omega = 0$ and the definition of the exterior derivative does not depend on the choice of reference system.

Since the equality

$$d\omega(X, Y) = \frac{1}{2} \{ X^\mu D_\mu \langle \omega, Y \rangle - Y^\mu D_\mu \langle \omega, X \rangle) - \langle \omega, [X, Y] \rangle \} \tag{A3}$$

where

$$[X, Y] = (X^\mu D_\mu Y^\nu - Y^\mu D_\mu X^\nu) D_\nu + X^\mu Y^\nu [D_\mu, D_\nu] \tag{A4}$$

and

$$[D_\mu, D_\nu] = \frac{1}{2} (\Phi_\nu \delta^\rho_\mu - \Phi_\mu \delta^\rho_\mu) D_\rho \tag{A5}$$

holds in the Lyra geometry for any 1-form, one has the structural equations

$$\frac{1}{2} T^\mu = de^\mu + \omega^\mu_\nu \wedge e^\nu \tag{A6}$$

$$\frac{1}{2} {}^{\mathrm{L}}R^\mu_{\nu\alpha\beta}\, e^\alpha \wedge e^\beta = d\omega^\mu_\nu + \omega^\mu_\alpha \wedge \omega^\alpha_\nu \tag{A7}$$

where the connection 1-form in a normal basis has the following form:

$$\omega^\mu_\nu = {}^{\mathrm{L}}\Gamma^\mu_{\nu\alpha} e^\alpha \tag{A8}$$

Making use of (A7) and (A8), one has

$${}^{\mathrm{L}}R^\alpha{}_{\beta\gamma\delta} = D_\gamma\, {}^{\mathrm{L}}\Gamma^\alpha_{\beta\delta} - D_\delta\, {}^{\mathrm{L}}\Gamma^\alpha_{\beta\gamma} + {}^{\mathrm{L}}\Gamma^\alpha_{\eta\gamma}\, {}^{\mathrm{L}}\Gamma^\eta_{\beta\delta} - {}^{\mathrm{L}}\Gamma^\alpha_{\eta\delta}\, {}^{\mathrm{L}}\Gamma^\eta_{\beta\gamma} + \frac{1}{2} {}^{\mathrm{L}}\Gamma^\alpha_{\beta\delta}\, \Phi_\gamma - \frac{1}{2} {}^{\mathrm{L}}\Gamma^\alpha_{\beta\gamma}\, \Phi_\delta \tag{A9}$$

This is exactly the Lyra curvature tensor given by equation (24).

From equations (A7) and (A8) one can easily derive the Bianchi identities:

$$^{\mathrm{L}}\nabla_{\epsilon}\ {}^{\mathrm{L}}R^{\alpha}_{\beta\gamma\delta} + {}^{\mathrm{L}}\nabla_{\delta}\ {}^{\mathrm{L}}R^{\alpha}_{\beta\epsilon\gamma} + {}^{\mathrm{L}}\nabla_{\gamma}\ {}^{\mathrm{L}}R^{\alpha}_{\beta\delta\epsilon} - {}^{\mathrm{L}}R^{\alpha}_{\beta\gamma\delta}\omega_{\epsilon} - {}^{\mathrm{L}}R^{\alpha}_{\beta\epsilon\gamma}\omega_{\delta} - {}^{\mathrm{L}}R^{\alpha}_{\beta\delta\epsilon}\omega_{\gamma} = 0 \tag{A10}$$

The Ricci tensor expressed in terms of the Riemannian connection and the displacement vector may be written as

$$\begin{aligned} {}^{\mathrm{L}}R_{\kappa\alpha} &= \frac{1}{\chi^2}\,{}^{\mathrm{R}}R_{\kappa\alpha} + \frac{1}{2\chi}(3\,{}^{\mathrm{R}}\nabla_{\alpha}\phi_{\kappa} - {}^{\mathrm{R}}\nabla_{\kappa}\phi_{\alpha} + g_{\kappa\alpha}\,{}^{\mathrm{R}}\nabla_{\beta}\phi^{\beta}) \\ &\quad + \frac{1}{4}(3\Phi_{\alpha}\phi_{\kappa} - \Phi_{\kappa}\phi_{\alpha} + g_{\kappa\alpha}\Phi^{\beta}\phi_{\beta}) + \frac{1}{2}(g_{\kappa\alpha}\phi_{\beta}\phi^{\beta} - \phi_{\alpha}\phi_{\beta}) \end{aligned} \tag{A11}$$

APPENDIX B

The gauge-invariant analog of the Einstein–Hilbert action on the Lyra manifold as proposed by Sen (1957) is given by the following integral:

$$S_G + S_M = \int {}^{\mathrm{L}}R\chi^4\sqrt{-g}\,d^4x + \int \Lambda\chi^4\sqrt{-g}\,d^4x \tag{B1}$$

where both ${}^{\mathrm{L}}R$ and Λ are Lyra scalars and $\chi^4\sqrt{-g}\,d^4x$ is an invariant measure.

The Sen equations that are valid in arbitrary gauge may be obtained from the variational principle by regarding the metric as the independent variable:

$$\frac{\delta}{\delta g^{\mu\nu}}(S_G + S_M) = 0 \tag{B2}$$

After necessary symmetrization they have the following form:

$$\begin{aligned} &{}^{\mathrm{R}}R_{\mu\nu} - \frac{1}{2}\,{}^{\mathrm{R}}Rg_{\mu\nu} + \chi({}^{\mathrm{R}}\nabla_{\nu}\Phi_{\mu} - g_{\mu\nu}\,{}^{\mathrm{R}}\nabla_{\sigma}\Phi^{\sigma}) + \frac{3}{2}\chi^2(\Phi_{\mu}\Phi_{\nu} - g_{\mu\nu}\Phi_{\sigma}\Phi^{\sigma}) \\ &\quad + \frac{3}{2}\chi^2(\phi_{\mu}\phi_{\nu} - \frac{1}{2}g_{\mu\nu}\phi_{\sigma}\phi^{\sigma}) - \frac{3}{2}\chi^2(\Phi_{\mu}\phi_{\nu} + \phi_{\mu}\Phi_{\nu} - g_{\mu\nu}\Phi_{\sigma}\phi^{\sigma}) = -8\pi\chi^2 T_{\mu\nu} \end{aligned} \tag{B3}$$

Making use of the formulas

$$\tilde{g}_{\mu\nu} = \lambda^{-2}g_{\mu\nu} \tag{B4}$$

$$\tilde{\phi}_{\mu} = \lambda^{\mu}\left(\phi_{\mu} + \frac{1}{\chi}\frac{\partial \ln \lambda^2}{\partial x^{\mu}}\right) \tag{B5}$$

$$\tilde{\Phi}_{\mu} = \lambda^{\mu}\left(\Phi_{\mu} + \frac{1}{\chi}\frac{\partial \ln \lambda^2}{\partial x^{\mu}}\right) \tag{B6}$$

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} + 3\lambda^{-2}g_{\mu\nu}\,{}^{\mathrm{R}}\nabla_{\beta}\lambda\,{}^{\mathrm{R}}\nabla^{\beta}\lambda - 2\lambda^{-1}g_{\mu\nu}\,{}^{\mathrm{R}}\nabla_{\beta}\,{}^{\mathrm{R}}\nabla^{\beta}\lambda - 2\lambda^{-1}\,{}^{\mathrm{R}}\nabla_{\mu}\,{}^{\mathrm{R}}\nabla_{\mu}\lambda \tag{B7}$$

one can easily obtain the Sen equations by transforming back the normal gauge Sen equation to the reference system with arbitrary gauge function. This means that the Sen equations are, as expected, doubly covariant.

The foregoing analysis indicates also that allowing variations with respect to the displacement vector that in the normal gauge results in the Einstein equations leads in arbitrary gauge to the conformally transformed Einstein field equations.

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